Contents

1 Representation Definitions

Why study representation theory? We would like to visualize difficult lie groups and algebras using techniques from linear algebra. First, let's talk representations generally then summarize some important notions regarding representations of Lie groups and algebras.

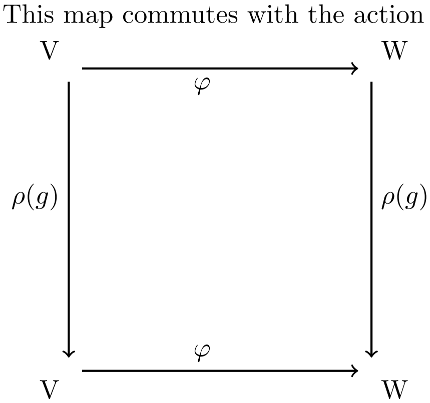
A **representation** of a group G is a vector space V and a homomorphism $\rho : G \to GL(V)$.

- $\rho(gw) = \rho(g)\rho(w)$
- For today, $V \cong \mathbb{C}^n$.
- Note that people usually denote $\rho(g)v$ as $g \cdot v$, where you're 'multiplying' a group element by a vector (via the corresponding linear transformation).

We call the Vector Space V a representation of G paired with the homomorphism ρ.

A homorphism between two representations V, W on the same group G is a linear map $\varphi : V \to W$.

- This map commutes with the action of $G:$ $\varphi(\rho(g)v) = \rho(g) \cdot \varphi(v)$



- The space of $G-$morphisms between V and W is denoted by $\text{Hom}_G(V, W)$.
- These homomorphisms between representations are often called 'intertwining operators,' which is what I'll call them.

Given some representation V of G, a **subrepresentation** is a vector subspace $V' \subset V$ where $\rho(g)v' \in V'$ for every $g \in G, v' \in V'$. This subspace is **invariant** since any matrix $\rho(g)$ acting on a vector $v' \in V'$ will still result in a vector in V'.

A non-zero representation V of group G is **irreducible** if its only subrepresentations are 0 and V.

V is **indecomposable** if $V \neq V_1 \oplus V_2$, where V_1 and V_2 are G representations.

Are 'indecomposability' and 'irreducibility' essentially equivalent? Not necessarily if the group in question is infinite. Take the following example:

$$\mathbb{Z} \to GL(2, \mathbb{R}),\ \rho(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The action on any $n \in \mathbb{Z}$ on vector $\begin{pmatrix} x \\ y \end{pmatrix}$ will be

$$\rho(1)^n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + ny \\ y \end{pmatrix}$$

Consider the unidimensional subspace $V_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$. Check to see if it is invariant:

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} c + 0 \\ 0 + 0 \end{pmatrix} \in V_1$$

So, this representation is reducible since there is exists a nontrivial invariant subspace V_1.

Can we decompose $\mathbb{R}^2$ as the direct sum of two invariant subspaces V_1 and V_2? Let V_2 to be the span of e_2:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin V_2$$

We cannot separate $\mathbb{R}^2$ into the direct sum of two invariant subspaces. Thus reducible but indecomposable.

As an important remark, these notions are equivalent for finite groups with our V of characteristic 0.

Before we show that, let us note that if V and W are $G-$representations, their direct sum $V \oplus W$ and their tensor product $V \otimes W$ will be representations given by the following:

$$V \oplus W:\ \rho(g)(v + w) = \rho(g)v + \rho(g)(w)$$

$$V \otimes W:\ \rho(g)(v \otimes w) = \rho(g)v \otimes \rho(g)w$$

$\text{Hom}_G(V, W)$ is also a $G-$rep:

$$(\rho(g)(\varphi))(v) = (g\varphi)(v) = g \cdot (\varphi(g^{-1} \cdot v)) = \rho(g) \cdot (\varphi(\rho(g^{-1}) \cdot v)$$

2 Intertwining operators and Schur's lemma

Theorem. Let V, W be irreducible complex reps of a finite group G.

(1) Then, either $V \cong W$, dim $\text{Hom}_G(V, W) = 1$, or

(2) dim $\text{Hom}_G(V, W) = 0$.

Proof. Take $\varphi \in \text{Hom}_g(V, W)$. Note that $\text{Ker}\varphi$ and $\text{Im}\varphi$ are **subrepresentations** of V.

$$\varphi(g \cdot v) = g\varphi(v) = g \cdot 0 = 0 \in \text{Ker}\varphi$$

$$g \cdot w = g \cdot \varphi(v) = \varphi(g \cdot v),\ g \cdot v \in V \implies g \cdot w \in \text{Im}\varphi$$

[Both possible by G-equivariance as established above]

If φ is nonzero, since V is irreducible then $\text{Ker}\varphi = 0$. $\text{Im}\varphi$ must be equal to W in that case too since W is also irreducible. We conclude that $V \cong W$.

What if $V = W$? Take λ, an eigenvalue of φ. Since we are with base field $\mathbb{C}$, then

$$\varphi - \lambda I \in \text{Hom}_G(V, W)$$

This has a nonzero kernel and we conclude that it is 0. $\varphi = \lambda I$ and thus it is an isomorphism. □

3 Maschke's Theorem

If G is finite ($V \cong \mathbb{C}^n$), then every indecomposable representation is an irreducible representation.

Proof. Let's start with the lemma that every unitary representation is completely reducible.

A complex representation V of a group G is called unitary if there is a G-invariant inner product: $(gv, gw) = (v, w)$, or equivalently $\rho(g) \in U(V)$ for any $g \in G$.

As a remark, this is equivalent to a positive definite Hermitian form:

1. Sesquilinear
2. Conjugate symmetry
3. Positive definite $(v, v) > 0$

We will do a proof by induction on dimension. Either V is irreducible and so we are done, or V has a subrepresentation W.

Then $V = W \oplus W^{\perp}$, where $W^{\perp}$ is the orthogonal component of the subspace W.

- Show that $W^{\perp}$ is a subrepresentation. If $w \in W^{\perp}$, then $(gw, v) = (w, g^{-1}v) = 0$ for any $v \in W$ (because $g^{-1}v \in W$). We conclude that $gw \in W^{\perp}$.

We continue this logic until we reach irreducible components such that

$$V = \oplus W_i$$

Now, it remains to be shown that a representation of a finite group is unitary.

Take some inner product $B(v, w)$ in V. We have no guarantee that it will be $G-$invariant, so instead we construct a new inner product $\tilde{B}$ as such:

$$\tilde{B}(v, w) = \frac{1}{|G|} \sum_{g \in G} B(gv, gw)$$

From this definition we determine that $\tilde{B}$ is positive definite as the sum of positive definite forms, and $G-$invariant with the following substitution: $gh = g'$.

$$\tilde{B}(hv, hw) = \frac{1}{|G|} \sum_{g \in G} B(ghv, ghw) = \frac{1}{|G|} \sum_{g' \in G} B(g'v, g'w)$$

Therefore this representation is unitary and we are done. □